$$f_3 = [\rho q^2 \sin(\beta - \theta)\partial\theta/\partial\beta + \cos(\beta - \theta)\partial p/\partial\beta]/D \times \sin^2(\beta - \theta)$$

$$f_4 = D^{-1}\rho q^2 \sin\theta \qquad D = M^2 - 1 - \cot^2(\beta - \theta)$$

where $\partial\theta/\partial\beta$ and $\partial p/\partial\beta$ can be written in terms of $\partial M_1/\partial\beta$ through the shock relations if the reference line is a shock.

The reference line need not be a shock. In the approach described in Ref. 6, the cowl surface is calculated rather than specified. In that case the reference line (initial data line in Ref. 6) might be a characteristic. Then Eqs. (4) and (5) could still be used with $\partial\theta/\partial\beta$ and $\partial p/\partial\beta$ calculated from the compatibility relations.

Although the tabulations of Ref. 1 are for $\gamma = 1.4$, the form of the gradients given in Eqs. (4) and (5) are not restricted to a constant γ gas. They apply to a flow with chemical reactions provided that chemically the flow is either frozen or in equilibrium. If the reference line is a shock, the appropriate shock relations are used to calculate the f_i. For nonequilibrium flow additional terms appear on the right-handside of Eqs. (4) and (5); these terms are proportional to the "production" terms in the rate equation. See Ref. 7 for the result with vibrational nonequilibrium included or Ref. 8 for a discussion of the general case with an arbitrary number of chemical reactions. The latter reference also discusses the use of other checks and the need for these in characteristic calculations.

In the previous discussion it was assumed that $r \neq 0$ since, for axisymmetric flow, special consideration of the axis is necessary. A kind of branch point singularity occurs there. The gradients and shock curvature can still be calculated for a given streamline (centerbody) curvature but ordinary differential equations must be solved. It is then a matter of individual judgement whether the effort to obtain this information is justified. The procedure, some results, and further references for getting this gradient information are given in Ref. 9.

Finally, gradients and shock curvature could be obtained at a point of shock reflection in the inlet (boundary-layer effects excluded). They would be obtained from Eqs. (4) and (5) using the nonuniform flow behind the incident shock to calculate the curvature of the reflected shock and surface gradients behind it.

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Energy State Approximation and Minimum-Fuel Fixed-Range Trajectories

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Introduction

THE energy state approximation[1] would appear to be a powerful tool for performance optimization. However, a dilemma appears in its application to minimum-fuel problems. Namely, for aircraft like the F4, the application of the Maximum Principle[2] fails to yield any solutions to the problem of a maximum range cruise. This will be shown to be a consequence of the fact that the velocity set[2] is not convex, allowing relaxed controllers that attain fuel economies superior to any control satisfying the Maximum Principle.

When an optimal control does fail to exist, there are suboptimal trajectories that achieve fuel economies superior to the full-powered climbs and zero throttle glides described in Ref. 1. These suboptimal trajectories contain a minimum fuel cruise segment, and achieve fuel economies very close to those of the optimum relaxed controller. Generally, it has been found that optimal controllers cannot exist when the aerodynamic data includes a classical minimum-fuel cruise point[3] within the flight envelope. They may exist otherwise, and then will correspond to that described in Ref. 1.

Finally, the climb path to the suboptimal cruise point may be well approximated by a Rutowski[4] minimum-fuel energy-climb path. This suggests that the minimum-fuel problem with range constraints can be well treated by combining minimum-fuel energy-climbs with classical cruises and maximum-range glides.

Energy State Equations

The energy state approximation assumes that altitude and velocity may be interchanged through zero-cost, constant-energy zooms and dives. The energy per unit weight, specific energy, is given by

$$E = V^2/2g + h \tag{1}$$

with V and h the velocity and altitude. The fuel rate $\dot{m}$ is assumed given by

$$\dot{m} = -\sigma T \tag{2}$$

where σ is the thrust specific fuel consumption and T the thrust. The energy state equations, with range x as the independent variable, are,[1]

$$dE/dx = [T(E,V,\pi) - D(E,V)]/W \tag{3}$$

$$dm/dx = -\sigma(E,V,\pi)T(E,V,\pi)/V \tag{4}$$

where π is the throttle angle, W is the (assumed constant) weight, and D is drag.

Drag, thrust and specific fuel consumption are indicated as functions of E and V rather than h and V to allow later use of V as a control variable. The limits on V are

$$Vs \leq V \leq (2gE)^{1/2} \tag{5}$$

where Vs is the stall velocity.

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Optimization Problem

Consider the following optimization problem. For

$$dE/dx = (T - D)/W \qquad E(0) = E_i, E(R) = E_f \quad (6)$$

minimize

$$C = \int_0^R (\sigma T/V)dx \quad (7)$$

with controls $0 \leq \pi \leq \pi_{max}$, $Vs \leq V \leq (2gE)^{1/2}$. The Hamiltonian[5] is given by

$$H = -\sigma T/V + \lambda(T - D)/W + \mu[V - (2gE)^{1/2}] \quad (8)$$

and the maximum principle states that if u^* is an optimal controller [i.e., $C(u^*) \leq C(u)$, all admissible u] there exists a nontrivial λ such that along the optimal trajectory,

$$H(E,\lambda,u^*) = \max_u H(E,\lambda,u) \quad (9)$$

with

$$d\lambda/dx = -\partial H/\partial E \quad (10)$$

Moreover, in this case

$$H(E,\lambda,u^*) = H_0 \quad (11)$$

where H_0 is a constant. Note that

$$\mu = 0 \text{ if } V < (2gE)^{1/2}$$

$$\mu \neq 0 \text{ if } V \geq (2gE)^{1/2} \quad (12)$$

From Eqs. (8) and (11) it can be shown that maximum H occurs either when $\pi = 0$ and drag is minimized with respect to velocity (i.e., maximum range glide), or when $\pi \neq 0$ at that π and V corresponding to maximum F, where

$$F = (T - D)/W|/[H_0 + (\sigma T/V)] \quad (13)$$

The transitions between powered and glide paths can occur only when

$$\max_{V,\pi\neq 0} F = -D_g/H_0W \quad (14)$$

where D_g is the glide drag. This follows from the continuity of λ along the optimal trajectory, and Eqs. (8) and (11). Note that the optimization problem has been reformulated as a single parameter optimization problem (i.e., select H_0 to satisfy terminal constraints).

The application of the maximum principle has assumed the existence of an optimal control. By examining certain geometric properties of the velocity set $\nu(E)$, defined below, it will be shown that an optimal control may not always exist. The velocity set is defined by

$$\nu_E = \left\{\frac{T - D}{W}, \frac{\sigma T}{V}\middle| \quad 0 \leq \pi \leq \pi_{max}, \quad Vs \leq V \leq (2gE)^{1/2}\right\} \quad (15)$$

with all quantities evaluated at a given energy level E. The minimum $\sigma T/V$ boundary of a typical set is shown (with exaggerated curvature) in Fig. 1.

Let point B correspond to the values of dE/dx and dm/dx achieved for π and V controls such that $[(T - D)/W + D_g/W]/\sigma T/V$ is maximized. For an $F4$, it is found that B occurs when $T > D$, at least over the energy range of interest. In any case, a line from $(0, -D_g/W)$ to point B forms a segment of the minimum $\sigma T/V$ boundary for the convex hull of ν_E. Values of dE/dx and dm/dx along this line may be achieved by chattering between point B and the glide point. The fraction of time spent at each point (or the probability with which each point is assumed) determine a convex combination of B and the glide point, thus yielding point C.

Now construct a line from $(-H_0,0)$ to point A, where F is a maximum. A point on a trajectory satisfying the maximum principle must be at A or be a maximum range glide.

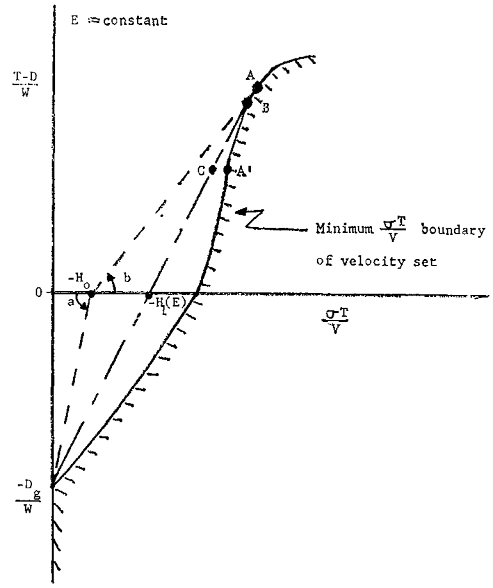

Fig. 1 Velocity set at energy level E.

Let $[-H_1(E),0]$ be the $T = D$ intercept of the minimum $\sigma T/V$ boundary of the convex hull of $\nu(E)$. Then, if $-H_0 > -H_1(E)$, a relaxed controller (e.g., C) could always achieve a lower $\sigma T/V$ value than would be given by the maximum principle (e.g., A'). Thus, along optimal trajectory segments, $-H_0 \leq -H_1(E)$. Draw a line from $(0,-D_g/W)$ to $(-H_0,0)$ on Fig. 1. Then the tangents of angles a and b correspond to the right-hand and left-hand sides, respectively, of Eq. (14). Then, from Eq. (14) it is seen that transitions between powered and unpowered flight may occur only when $H_0 = H_1(E)$.

Finally, assume that $-H_1(E)$ attains a minimum within the flight envelope at energy level E_m. That this does occur, has been verified for the $F4$ (and for aircraft number 1 in Ref. 1). In fact, the energy level E_m is approximately that where $\sigma D/V$ achieves a minimum.

The previous development will now be applied to a specific problem, and the lack of existence of an optimal control for that problem will be established. Consider the minimum fuel problem with $E_i = E_f = E_m$, $R > 0$. Any admissible (i.e., satisfying maximum principle) variable energy trajectory must have a climb path at level E_m, and if optimal, $-H_0 \leq -H_1(E_m)$. But then when $E \neq E_m$,

$$-H_0 \leq -H_1(E_m) < H_1(E) \quad (16)$$

But then no transitions between powered and glide flight segments are permissible and the terminal energy constraint is not satisfied. Thus, there do not exist optimal controls for this problem. There is, of course, an optimum relaxed controller. That controller achieves

$$dm/dx = +H_1(Em) \text{ and } dE/dx = 0$$

Limitations of the Energy State Approximation

The nonexistence of an optimal control for the above problem is symptomatic of a more general problem. Namely, when the minimum $\sigma T/V$ boundary of $\nu(E)$ and its convex hull do not correspond, there may exist optimum relaxed controllers whose performance cannot be achieved by a classical control. The fact that drag is nonlinear in the control variable V makes this a likely phenomenom.

Thus, any results stemming from application of the maximum principle (or any similar variational approach) warrant very careful scrutiny. The results in Ref. 1 can thus be true only if the minimum $\sigma T/V$ boundary of $\nu(E)$ is convex, or if $-H_1(E)$ does not take on a minimum within the flight envelope. This is a nontypical occurrence.

More often, it will result that fixed-range minimum-fuel trajectories fall into two categories. Those problems for which the range is sufficiently small that energy level Em is never attained [i.e., $-H_0 > -H_1(E_m)$], yield optimal trajectories consisting of a single powered climb and a maximum range glide. All other minimum-fuel trajectories consist of a climb path with $H_0 = H_1(E_m)$, a relaxed cruise of appropriate length at energy level E_m, and a maximum range glide.

Suboptimal Solutions

The optimum relaxed controller for the cruise segment achieves specific range $[-H_1(E_m)]^{-1}$. Rather than seeking controllers approximating the relaxed controller, one may select a suboptimal solution approximating the relaxed controller's performance.

Evaluation of aerodynamic data for the $F4$ yields a trivial difference between $-H_1(E_m)$ and the minimum value of $\sigma D/V$, as well as small differences in the energy levels at which $H_1(E)$ and $\sigma D/V$ are minimized. Thus, the energy state approximations support a classical cruise. In fact, investigations to date indicate that a cruise is superior to a full throttle climb and glide trajectory, whenever $\sigma D/V$ has a minimum within the flight envelope.

Secondly, from Fig. 1, for $0 \leq -H_0 \leq -H_1(E)$, point A does not vary significantly. Therefore, a Rutowski[4] minimum-fuel energy-climb path (i.e., $H_0 = 0$) is usually acceptable for the climb segment. Therefore, suboptimal trajectories using Rutowski climb paths, classical minimum cruise fuel cruises, and maximum range glides are also suggested.

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Sonic Boom Reflection Factors for Flight near the Threshold Mach Number

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When an aircraft travels at a Mach number slightly greater than one, the sonic boom pressure wave which is generated by the aircraft is often unable to reach the ground due to atmospheric refraction. The maximum flight Mach number for which this will occur is called the threshold Mach number. The threshold Mach number depends upon the flight altitude and atmospheric conditions. For flight above 36,000 ft in the standard atmosphere the threshold Mach number is about 1.15. When an aircraft travels slightly faster than the threshold Mach number the wavefront of the sonic boom wave is nearly perpendicular to the ground where it strikes the ground. The magnitude of the sonic boom which is observed on the ground depends not only upon the amplitude of the incident pressure wave but also upon the manner in which the wave reflects from the ground. Because the sonic boom pressure wave is very weak it is usually assumed in sonic boom calculations that the amplitude of the pressure disturbance on the ground is, at most, twice the amplitude of the incoming wave; that is, the sonic boom reflection factor is assumed to have a maximum of two.

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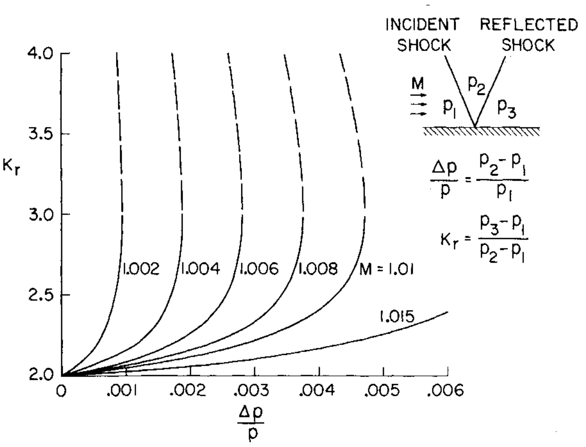

Fig. 1 Regular reflection of a very weak shock wave from a smooth surface.

The purpose of this Note is to consider the reflection of a very weak shock wave off a smooth surface, for the condition in which the incident shock wave is nearly perpendicular to the surface. It is found that in this situation the pressure rise Δp across the reflected shock can be up to twice the pressure rise across the incident shock, indicating that sonic boom reflection factors as large as three are possible for aircraft traveling near the threshold Mach number. Commercial transport aircraft which cruise slightly slower than the threshold Mach number are now under serious consideration for overland routes. Of course, if sonic cutoff does occur then the problem of reflection of the sonic boom wave does not arise. However, a change in the atmospheric conditions or in the elevation of the ground might cause the sonic boom pressure wave to reach the ground, and it is in this situation that a sonic boom reflection factor greater than two is possible.

The reflection of a very weak shock wave off a smooth surface, when the incident shock is nearly perpendicular to the surface, has been investigated using the oblique shock relations of Ref. 1 (Eqs. 128, 132, 139, and 150). Figure 1 shows the resulting relationship between the reflection factor (K_r), flow Mach number (M), and incident shock wave strength ($\Delta p/p$). It is seen that for values of $\Delta p/p$ typical of the sonic boom front shock and values of M close to one, the reflection factor can range between two and three. For any given $\Delta p/p$ and M there are two reflected shocks that are mathematically possible. However, the stronger solution, shown by dashed lines, does not occur in real life. Figure 1 is not intended to be used quantitatively in sonic boom calculations because $\Delta p/p$ cannot be estimated theoretically when the flight Mach number is very near the threshold Mach number. However, Fig. 1 does show that the sonic boom reflection factor should not be automatically assumed to be less than or equal to two, just because the sonic boom pressure wave is very weak. It is seen in Fig. 1 that for each Mach number there is a maximum $\Delta p/p$ for which a regular reflection is possible. If $\Delta p/p$ is larger than this maximum, then a Mach reflection occurs. The flow for this type of reflection is very complex due to the curvature of the shocks and the subsonic region downstream of the Mach stem. At present, theoretical methods are inadequate for predicting quantitative properties of the Mach reflection when the shocks are weak.

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